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On the Dirac approach to constrained dissipative dynamics

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Abstract

In this paper, we propose a novel algebraic and geometric description for the dissipative dynamics. Our formulation bears some similarity to the Poisson structure for non-dissipative systems. We develop a canonical description for constrained dissipative systems through an extension of the Dirac brackets concept, and we present a new formula for calculating Dirac brackets. This formula is particularly useful in the description of dynamical systems with many second-class constraints. After presenting the necessary formal background we illustrate our method on several examples taken from particle dynamics, continuum media physics and wave mechanics.

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1. Introduction

The first systematic attempt to provide a mathematically consistent quantization procedure for constrained systems was made by Dirac [1], who derived a formal 'replacement' for the canonical Poisson brackets, which today plays a fundamental role in the canonical formalism for the constrained Hamiltonian systems on both classical and quantum levels. In spite of the considerable attention paid to this formula in the mathematical literature [2–4] and several attempts to use the Dirac brackets in the quantization of gauge invariant systems [5, 6], until recently there were few attempts to actually use this formalism in more conventional applications. We have recently provided a few examples of these applications in classical and continuum mechanics [7].

The canonical formalism applies to conservative systems. These systems form a relatively small sub-class of interesting physical systems, since most of the other systems describing this phenomenon at some effective rather than fundamental level are dissipative. In the past, several attempts were made to describe dissipative classical mechanics in a manner similar to its canonical description. One of these attempts, i.e. the so-called metriplectic approach [8, 9], was advocated as a natural extension of the *mixed canonical–dissipative dynamic* proposed by Enz [10]. In this paper, we extend further the concept of the metriplectic dynamics to what we shall call *semimetric–Poissonian dynamics*, which is nothing but a natural combination of semimetric dynamics (a dissipative part) and Poissonian dynamics (a conservative part). We propose a canonical description for the constrained dissipative systems through an extension of the concept of Dirac brackets [1] developed originally for conservative constrained Hamiltonian dynamics, to the non-Hamiltonian, namely *metric* and mixed *metriplectic*, constrained dynamics. It turns out that this generalized unified formula for the Dirac brackets is very useful in the description and analysis of a wider class of dynamical systems. To proceed with our approach we develop a new formula for calculating Dirac brackets which is particularly effective in finding equations of motion and constants of motion for systems with many constraints.

In order to make this paper self-contained we include in section 2 a short 'primer' to the Poisson geometry, dynamics and Dirac brackets.

The rest of the paper is organized into four sections. In section 3 we discuss semimetric dynamics and some elementary features of the related mathematical structures such as *semimetric algebras* and *SJ identity*, the latter should be regarded as a dynamical symmetric version of the Jacobi identity. We develop symmetric concepts in analogy with those in the Poisson category in the appendix.

In section 4 we discuss semimetric dynamics subject to some constraints. We derive a symmetric analogue of the Dirac brackets and provide its geometric interpretation as an induced metric on some submanifolds of a Riemannian manifold. We also present a new effective algorithm for calculation of the Dirac brackets in both symmetric and antisymmetric cases. A few examples, for finite and infinite dimensional cases, are also discussed in some detail.

In section 5, we discuss semimetric–Poissonian dynamics which is a combination of semimetric dynamics, discussed in section 3, and Poissonian dynamics mentioned in section 2. This section also includes the extended Dirac approach to constrained semimetric–Poissonian dynamics, which is a combination of the Dirac approach to constrained semimetric dynamics discussed in section 4 and the usual approach to constrained Poissonian dynamics.

In section 6, we discuss some interesting physical examples: the dissipative formulation of the Schrödinger equation due to Gisin [11], the Landau–Lifshitz–Gilbert equations for damped spins [8, 12] and the dynamics of a damped body—including among others the Morrison equation of a damped rigid body [9]. We also discuss a novel description of the incompressible viscous fluid.

Some important but not crucial mathematical aspects of the extended canonical formalism for constrained metric and mixed semimetric–Poissonian systems will be given in a forthcoming publication [13]. Computational aspects of the Dirac brackets—symmetric, antisymmetric or mixed—will be published shortly [14].

2. Poisson geometry, dynamics and Dirac brackets

From the algebraic point of view the *Poisson algebra* is a linear space $\mathcal{F}$ equipped with two structures:

(i) the commutative algebra structure with the (associate) multiplication $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$; the product of two elements f, g is denoted simply by fg,
(ii) the Lie algebra structure with the Lie bracket $\{\cdot, \cdot\}$: $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$; the product of two elements f, g is denoted by $\{f, g\}$,

related to each other by the Leibniz rule:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g. \tag{2.1}$$

The Lie bracket of a Poisson algebra is called the *Poisson bracket*.

The *Poisson manifold* is a smooth manifold M for which the commutative algebra of smooth functions on M, $C^\infty(M)$ is equipped with the Poisson bracket. The Poisson bracket $\{\cdot, \cdot\}$ acts on each function as a derivation, thus there exists a contravariant (2, 0)-tensor Π such that $\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g)$ for every functions f, g. In the local coordinates (z^k)

$$\{f, g\}(z) = \sum_{i,j}^{m} \Pi^{ij}(z)\partial_i f \partial_j g \quad \partial_k \equiv \frac{\partial}{\partial z^k}\,. \tag{2.2}$$

The tensor Π which defines a Poisson bracket is called a *Poisson tensor*. The antisymmetry of a Poisson bracket implies that the tensor Π must be antisymmetric, so $\Pi^{ij} = -\Pi^{ji}$. The Jacobi identity requires that

$$\sum_{l=1}^{N} \Pi^{li}\partial_l \Pi^{jk} + \Pi^{lj}\partial_l \Pi^{ki} + \Pi^{lk}\partial_l \Pi^{ij} = 0. \tag{2.3}$$

A *Hamiltonian vector field* generated by a function h is a vector field defined by $X_h(f) = \{f, h\}$ for every function f. All flows generated by the Hamiltonian vector fields—*the Hamiltonian flows*, preserve the Poisson structure.

Poisson dynamics or *generalized Hamiltonian dynamics* is a dynamics generated by some Hamiltonian vector field, for which the Hamiltonian function plays a physically specific role.

Let TM and T^*M denote the tangent and cotangent bundle of the manifold M. The Poisson tensor Π induces a bundle map $\Pi^\sharp$: $T^*M \to TM$ which is defined by $\Pi^\sharp(\mathrm{d}f) := X_f$ for all functions f. The *rank* of a Poisson structure at point z is defined to be the rank of $\Pi^\sharp_z$: $T^*_zM \to T_zM$, which is equal to the rank of matrix $\Pi^{ij}(z)$ in the local coordinates (z^k). A Poisson structure with constant rank equal to the dimension of the manifold M is called *symplectic* or *nondegenerate*. In this case the 'inverse map' of Π, denoted by ω, is a symplectic 2-form and $\omega(X_f, X_g) = \{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g)$. The Darboux theorem states that for every symplectic structure there exists, locally, a canonical coordinate system $(x_1, \ldots, x_k, p_1, \ldots, p_k)$ such that $\Pi = \frac{\partial}{\partial \boldsymbol{x}} \wedge \frac{\partial}{\partial \boldsymbol{p}}$, $(\omega = \mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{p})$, or equivalently, $\{x_i, x_j\} = \{p_i, p_j\} = 0, \{x_i, p_j\} = \delta_{ij}$.

The invariance of the Poisson structure under the Hamiltonian flows implies the constancy of the tensor Π rank along the orbits of such flows. The orbit of each point of M under the action of all Hamiltonian flows forms a symplectic manifold called a *symplectic leaf*. Since M is a union of such orbits, every Poisson manifold is a smooth union of disjoint connected symplectic manifold (symplectic leaves) of various ranks.

The splitting theorem [15] for Poisson manifolds states that locally every Poisson manifold is the product of a symplectic manifold and a Poisson manifold with zero rank. In other words, locally in the neighbourhood of the point $\boldsymbol{z}_0$ there always exists a canonical coordinate system: $(x_1, \ldots, x_k, p_1, \ldots, p_k, z_{2k+1}, \ldots, z_n)$ such that $\{x_i, x_j\} = \{p_i, p_j\} = 0, \{x_i, p_j\} = \delta_{ij}, \{x_i, z_l\} = \{p_j, z_l\} = 0, \{z_r, z_s\} = A_{rs}$ and $A_{rs}(\boldsymbol{z}_0) = 0$.

The map φ: $M_1 \to M_2$ between two Poisson manifolds is called a *Poisson mapping* iff $\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi$. The *Poisson mapping* is a natural generalization of the well-known classical mechanics notion of the *canonical transformation*.

In the usual formulation of classical mechanics the constrained dynamics can be visualized geometrically as the dynamics on some submanifold of the system phase space. Similarly, the constrained Poisson dynamics can be represented as such on some submanifold of the Poisson manifold. However, it is not always possible to define induced Poisson structure on a submanifold and therefore we have no obvious way to generalize constrained Poisson dynamics. If the Poisson structure is non-degenerate (symplectic case) then on each submanifold there exists an induced 2-form which becomes symplectic if it is non-degenerate. Further, for an arbitrary submanifold N of a symplectic manifold (M, ω) there always exists a maximal submanifold $N' \subset N$ such that $\omega_{|N'}$ is non-degenerate, then N' has an induced symplectic, hence Poisson, structure.

Dirac [1] has proposed an algebraic procedure to deal with the constrained dynamics. First, consider any finite set of (linearly independent) constraints $\mathcal{A} = \{a_1, \ldots, a_k\} \subset \mathcal{F} \equiv C^\infty(M)$ and define weakly vanishing (or weakly zero) elements as linear combinations of constraints with arbitrary function coefficients, i.e. $\mathcal{I} = \left\{\sum_i f_i a_i, \text{ where } f_i \in \mathcal{F}, a_i \in \mathcal{A}\right\}$.

Element $f \in \mathcal{F}$ is called *first-class* with respect to the set of constraints $\mathcal{A}$ iff it has weakly zero bracket with all constraints, i.e. $\forall a \in \mathcal{A}\colon \{f, a\} \in \mathcal{I}$. Otherwise, it is called *second-class*. The set of all first-class elements denoted by $\mathcal{F}_1(\mathcal{A})$, forms a linear subspace of $\mathcal{F}$ and the set of all second-class elements denoted by $\mathcal{F}_2(\mathcal{A})$.

This classification divides the set of constraints into two subsets: first-class constraints $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{F}_1(\mathcal{A})$ and second-class constraints $\mathcal{A}_2 = \mathcal{A} \cap \mathcal{F}_2(\mathcal{A})$. The number of second-class constraints must be even $\mathcal{A}_2 = \{\Theta_1, \ldots, \Theta_{2s}\}$. Dirac has proved that the Gramm matrix of second-class constraints, $[\{\Theta_i, \Theta_j\}] = W$, is weakly non-degenerate. This allowed him to define a new antisymmetric Leibniz bracket, known as the *Dirac bracket*:

$$\{f, g\}_{\mathrm{D}} = \{f, g\} - \sum_{i,j=1}^{2s} \{f, \Theta_i\} C_{ij} \{\Theta_j, g\} \tag{2.4}$$

where $C = [C_{ij}] = W^{-1}$ is an inverse matrix of W. Using (2.4) one can check that the Dirac brackets possess all the required properties of the Poisson brackets. The (algebraic) proof of the Jacobi identity is difficult. One can also check that all the second-class constraints are Casimirs with respect to the Dirac bracket, i.e. $\{\Theta_k, f\}_{\mathrm{D}} = 0$ for all f.

3. Semimetric manifolds and semimetric dynamical systems

In this section, we introduce a concept of *semimetric algebras* and *semimetric manifolds* which play a similar role in the description of dissipative systems as the Poisson algebras and the Poisson manifolds in the description of conservative dynamics. In the last two sections of this work, we will show that the *semimetric structure* together with the *Poisson structure* are sufficient for 'canonical' description of a wide class of dissipative dynamical systems.

Let X be a non-empty set. A *semimetric bracket* on the linear space of real functions defined on X, namely $\mathcal{F} = Fun(X)$, is a bilinear operation $\prec\cdot, \cdot\succ\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ which satisfies the following requirements:

(i) It is symmetric: $\forall f, g \in \mathcal{F}\colon \prec f, g\succ = \prec g, f\succ$.
(ii) It satisfies the Leibniz rule: $\forall f, g, h \in \mathcal{F}\colon \prec fg, h\succ = \prec f, h\succ g + f \prec g, h\succ$.
(iii) It is non-negative definite: $\forall f \in \mathcal{F}\colon \prec f, f\succ \geqslant 0$, i.e. the function $\prec f, f\succ$ is a non-negative definite function, $\forall x \in X\colon \prec f, f\succ(x) \geqslant 0$.

Note that if a bilinear operation on $\mathcal{F}$ satisfies conditions (i) and (ii), it is called a *pseudo-metric bracket* or *symmetric Leibniz bracket*. A symmetric Leibniz bracket which satisfies the condition (iii*) positive definite: $\forall f \in \mathcal{F}$: $\prec f, f\succ \geqslant 0$ and $\prec f, f\succ = 0$ iff $f = \text{const}$ (at least locally),

is called a *metric bracket*.

A *symmetric Leibniz algebra* is a linear space $\mathcal{F}$ equipped with two structures: commutative algebra structure with the (associate, commutative) multiplication and symmetric (non-associate, non-commutative) structure and these two structures are related by the Leibniz rule. A *semimetric algebra* is a symmetric Leibniz algebra whose bracket is semimetric.

As we shall see the semimetric algebra can be used to describe the wide class of dissipative classical systems akin to the description of the non-dissipative dynamics by means of the Poisson algebra.

From now on we assume that X is a smooth finite dimensional manifold, namely *phase space*, and $\mathcal{F} = C^\infty(X)$ is a space of all smooth functions on X.

There is a one-to-one correspondence[1] between the symmetric Leibniz brackets on the space of functions and the symmetric tensors on the manifold X : $\prec f, g\succ = G(\mathrm{d}f, \mathrm{d}g)$, where G is a contravariant tensor field of the type $(2, 0)$ on X. In local coordinates (z^k), each symmetric tensor is of the form $G(z) = \sum_{i,j} G^{ij}(z)\frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j}$, where $G^{ij} = G^{ji}$, hence each symmetric Leibniz bracket locally must be of the form:

$$\prec f, g\succ(z) = \sum_{i,j=1}^{N} G^{ij}(z)\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j} \qquad f, g \in C^\infty(X). \tag{3.1}$$

A symmetric bracket becomes a semimetric bracket iff it is non-negative, i.e. the matrix $[G^{ij}]$ is non-negative definite.

A semimetric bracket is called a *metric bracket* if G is positive definite (non-negative and non-degenerate), i.e. with constant maximal rank, rank $G = \dim X$. In general, the tensor G may have a non-constant rank which depends on points.

Definition 3.1. *A semimetric manifold is a smooth manifold M for which the commutative algebra of smooth functions on M is a semimetric algebra, i.e. it is equipped with a semimetric bracket. Geometrically, a semimetric manifold can be viewed as a pair* (M, G) *where G is a semimetric tensor (or cometric tensor), i.e. symmetric, non-negative definite:* $G(\mathrm{d}f, \mathrm{d}f) \geqslant 0$ *for every function f, contravariant* $(2, 0)$*-tensor field.*

Example 3.1. Tensor $G(z) = \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i} - \sum_{j=k+1}^{n} \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_j}$ is symmetric and non-degenerate, but it is not non-negative definite. Tensor $G(z) = \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i}$ for $k < n$, and tensor $G(z) = z_1^2 \frac{\partial}{\partial z_1} \otimes \frac{\partial}{\partial z_1} + \left(z_2^2 z_3^2\right) \frac{\partial}{\partial z_2} \otimes \frac{\partial}{\partial z_2} + \sum_{i=3}^{n} z_i^2 \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i}$ are non-negative definite, but degenerate.

A *dissipative vector field* generated by the function h is a vector field defined by $X_h^{\mathrm{D}}(f) = \prec f, h\succ$ for all functions f. The flow generated by a dissipative vector field should be called a *dissipative flow*. Dissipative flows essentially differ from the Hamiltonian counterpart; they do not preserve the symmetric structure.

If the $(2, 0)$-tensor G is positive definite, there exists a symmetric $(0, 2)$-tensor $\mathcal{G}$, its inverse, such that $G(\mathrm{d}f, \mathrm{d}h) = \mathcal{G}\left(X_f^{\mathrm{D}}, X_h^{\mathrm{D}}\right)$, which is exactly a Riemannian metric tensor. In the local coordinates (z_k), if $G(z) = \sum_{i,j} G^{ij}(z)\frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j}$, the tensor $\mathcal{G}(z) = \sum_{i,j} G_{ij}(z)\mathrm{d}z_i \otimes \mathrm{d}z_j$ where $\sum_j G^{ij} G_{jk} = \delta_k^i$.

The concept of semimetric manifold is then a natural generalization concept of the Riemann manifold. It is analogous to a generalization from a symplectic manifold to the Poisson manifold.

[1] Note that this is true for smooth functions $C^\infty(X)$, but false for $C^k(X)$.

Similarly, as in the Poissonian case, it is not always possible to define induced semimetric structure on a submanifold, hence there is no obvious way to describe canonically constrained semimetric dynamics.

The map $F\colon M_1 \to M_2$ between two semimetric manifolds is called a *semimetric mapping* iff it maps the semimetric structures, i.e. $\prec f \circ F, g \circ F \succ_1 = \prec f, g \succ_2 \circ F$.

Proposition 3.1. *Let* $(\mathcal{F}, \cdot, \prec \cdot, \cdot \succ)$ *be a semimetric algebra.*

(a) *The Schwartz inequality holds*

$$\forall f, g \in \mathcal{F}\colon \prec f, f \succ \prec g, g \succ \geqslant \prec f, g \succ^2. \tag{3.2}$$

(b) *Let* $f_1, f_2, \ldots, f_n$ *be arbitrary elements of* $\mathcal{F}$. *Then the square matrix*

$$Gram(f_1, \ldots, f_n) = \begin{bmatrix} \prec f_1, f_1 \succ & \ldots & \prec f_1, f_n \succ \\ \prec f_2, f_1 \succ & \ldots & \prec f_2, f_n \succ \\ \ldots & \ldots & \ldots \\ \prec f_n, f_1 \succ & \ldots & \prec f_n, f_n \succ \end{bmatrix} \tag{3.3}$$

is non-negative definite. In particular, $\det Gram(f_1, \ldots, f_n) \geqslant 0$. *Furthermore, if* $\mathcal{F}$ *is a metric algebra, then* $\det Gram(f_1, \ldots, f_n) = 0$ *iff* $\{f_i\}_{i=1}^n$ *are affine linear dependent.*

Proof.

(a) Indeed, for each real number $\lambda \in \mathbf{R}$ the expression $0 \leqslant \prec f - \lambda g, f - \lambda g \succ = \lambda^2 \prec g, g \succ - 2\lambda \prec f, g \succ + \prec f, f \succ$ is a non-negative quadratic form in the real number λ. Hence, the discriminant $\triangle = 4 \prec f, g \succ^2 - 4 \prec f, f \succ \prec g, g \succ \leqslant 0$ must be non-negative.

If $\mathcal{F}$ is a metric algebra, then $\prec f, g \succ^2 = \prec f, f \succ \prec g, g \succ$ iff f, g are affine linear dependent, i.e. $f - \lambda g = \text{const}$.

(b) For each vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$, denote $\boldsymbol{a} \cdot \boldsymbol{f} = \sum_i a_i f_i$, we have $\boldsymbol{a}^T Gram(f_1, \ldots, f_n) \boldsymbol{a} = \prec \boldsymbol{a} \cdot \boldsymbol{f}, \boldsymbol{a} \cdot \boldsymbol{f} \succ \geqslant 0$. □

Example 3.2.

(1) The natural Euclidean metric of n-dimensional Euclidean space $X = \mathbf{R}^n$ induces a natural metric structure on $C^\infty(X)$

$$\prec f, g \succ (\boldsymbol{x}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \qquad \text{where} \quad f, g \in C^\infty(\mathbf{R}^n) \quad \boldsymbol{x} \in \mathbf{R}^n. \tag{3.4}$$

We shall call the metric bracket (3.4) the Euclidean metric bracket.

(2) Here is a simple but general construction of the semimetric structure in the space of smooth functionals over a Hilbert space. Let X be a Hilbert space with a scalar product $\langle \cdot | \cdot \rangle$ and A be a linear operator on X. One can define a semimetric structure on the space of all smooth functionals over X as follows:

$$\prec \Phi, \Psi \succ (\boldsymbol{f}) = \left\langle A \frac{\delta \Phi}{\delta \boldsymbol{f}} \middle| A \frac{\delta \Psi}{\delta \boldsymbol{f}} \right\rangle. \tag{3.5}$$

For instance, let X be a Hilbert space of functions $X = \{\boldsymbol{f}\colon \mathbf{R}^n \to \mathbf{R}^d\}$ with a scalar product

$$(\boldsymbol{f} | \boldsymbol{g}) = \sum_{i,j=1}^d \int \mathrm{d}^n \boldsymbol{x}\, f_i(\boldsymbol{x}) G^{ij}(\boldsymbol{x}) g_j(\boldsymbol{x})$$
$$\text{where} \quad \boldsymbol{f} = (f_1, f_2, \ldots, f_d) \quad \boldsymbol{g} = (g_1, g_2, \ldots, g_d) \in X. \tag{3.6}$$

The above scalar product on X defines a semimetric bracket on the space of smooth functionals over X by

$$\prec\Phi, \Psi\succ(\boldsymbol{f}) = \sum_{i,j=1}^{d} \int \mathrm{d}^n \boldsymbol{x} \frac{\delta\Phi}{\delta f_i(\boldsymbol{x})} G^{ij}(\boldsymbol{x}) \frac{\delta\Psi}{\delta f_j(\boldsymbol{x})} \tag{3.7}$$

where $\Phi, \Psi \in C^\infty(X)$.

To be specific, let $n = d$ and G be a differential operator of the form D^+D, for instance, let $G^{ij}(\boldsymbol{x}) = -\left[a\frac{\partial^2}{\partial x_i \partial x_j} + b\delta_{ij}\triangle\right]$, where $a, b \geqslant 0$ and $\triangle$ denotes the Laplace operator. The symmetric bracket (3.7) now assumes the form

$$\prec\Phi, \Psi\succ(\boldsymbol{f}) = \int \mathrm{d}^n \boldsymbol{x} \left\{ a \left[\nabla \cdot \frac{\delta\Phi}{\delta \boldsymbol{f}(\boldsymbol{x})}\right] \left[\nabla \cdot \frac{\delta\Psi}{\delta \boldsymbol{f}(\boldsymbol{x})}\right] + b \sum_i \left[\nabla \frac{\delta\Phi}{\delta f_i(\boldsymbol{x})}\right] \cdot \left[\nabla \frac{\delta\Psi}{\delta f_i(\boldsymbol{x})}\right] \right\}. \tag{3.8}$$

As we shall see this is exactly a semimetric bracket as needed in viscous fluid dynamics.

We shall call *semimetric dynamics* a dynamics which is governed by some semimetric (dissipative) vector field. In the local coordinate system (z_k), we have the following system of first-order differential equations:

$$\dot{z}^i = \prec z^i, \mathcal{S}\succ = X^{\mathrm{D}}_{\mathcal{S}}(z^i) = \sum_{j=1}^{N} G^{ij}(z) \frac{\partial \mathcal{S}}{\partial z^j} \qquad i, j = 1, 2, \ldots, N \tag{3.9}$$

where $\prec\cdot, \cdot\succ$ is a semimetric bracket, $X^{\mathrm{D}}_{\mathcal{S}}$ is a dissipative vector field generated by function $\mathcal{S}$ defined on the phase space and G is a semimetric tensor. In some very special cases, the function $\mathcal{S}$ has a physical interpretation as entropy. $\mathcal{S}$ is always non-decreasing, since

$$\dot{\mathcal{S}} = \prec\mathcal{S}, \mathcal{S}\succ \geqslant 0. \tag{3.10}$$

If f, g, are constants for semimetric dynamics, then the Leibniz rule ensures that fg is also a constant, but $\prec f, g\succ$ is usually not a constant due to the lack of the Jacobi identity. We shall now introduce a new concept, a symmetric analogy of the Jacobi identity, which we call the *SJ identity*.

(iv) *SJ identity*:

$$\forall f, g, h \in \mathcal{F}\colon 2\prec\prec f, g\succ, h\succ = [\prec\prec f, h\succ, g\succ + \prec\prec g, h\succ, f\succ]$$
$$\Longleftrightarrow \forall f, h \in \mathcal{F}\colon \prec\prec f, f\succ, h\succ = \prec\prec f, h\succ, f\succ. \tag{3.11}$$

A symmetric dynamical system is called a *SP dynamics* ('symmetric–Poisson dynamics') iff the symmetric Jacobi identity (3.11) holds.

Proposition 3.2. *If f, g are constants of the SP dynamics, then $\prec f, g\succ$ is also a constant of motion. In particular, if f is a constant, then $\prec f, f\succ$ also.*

Proof. %%%Indeed, since $\prec f, \mathcal{S}\succ = \prec g, \mathcal{S}\succ = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\prec f, g\succ = \prec\prec f, g\succ, \mathcal{S}\succ = \frac{1}{2}[\prec\prec f, \mathcal{S}\succ, g\succ + \prec\prec g, \mathcal{S}\succ, f\succ] = 0. \tag{3.12}$$

□

Therefore, constants of SP dynamics form a subalgebra. This property is quite useful in finding constants of motion for SP dynamics.

Note that each autonomous dynamical system described by the system of the first-order differential equations

$$\dot{x}_k = F_k(x_1, \ldots, x_n) \qquad k = 1, \ldots, n \tag{3.13}$$

is a pseudo-metric system. Indeed, one can choose as the function $\mathcal{S}$, $\mathcal{S} = \sum_{k=1}^{n} x_k$, and as a diagonal pseudo-metric $G^{ij}(x) = \delta^{ij} F_i(x)$.

Furthermore, locally and almost everywhere each dynamical system (3.13) is metric. Indeed, for each point of the set $\{\boldsymbol{x}: \forall k = 1, \ldots, n: F_k(\boldsymbol{x}) \neq 0\}$ there exists a neighbourhood U such that functions F_k do not change their sign inside U. Let us denote $s_k = \operatorname{sign} F_k = \pm 1$ in U. In the neighbourhood U, the system should be regarded as a metric system with, for instance, $G^{ij}(x) = s_i \delta^{ij} F_i(x) \geqslant 0$ and $\mathcal{S} = \sum_k s_k x_k$. In particular, locally and almost everywhere, Poisson dynamics also admits a metric description. Each dynamical system of the type $\dot{x}_k = F_k(\boldsymbol{x})$ should be regarded as the Poissonian system after doubling the number of variables. Indeed, consider a canonical Poisson structure: $\{x_i, x_j\} = 0 = \{p_i, p_j\}$, $\{x_i, p_j\} = \delta_{ij}$ and let $\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}) = \sum_k p_k F_k(\boldsymbol{x})$, the canonical equations follow:

$$\dot{x}_k = \{x_k, \mathcal{H}\} = F_k(\boldsymbol{x}) \quad \dot{p}_k = \{p_k, \mathcal{H}\} = -\sum_j p_j \frac{\partial F_j(\boldsymbol{x})}{\partial x_k}. \tag{3.14}$$

A system is *non-Poissonian* (or *non-metric*) if it cannot be written in the Poisson (resp. metric) form without changing the number of variables. Each dynamical system of the type $\dot{x}_k = F_k(\boldsymbol{x})$, where functions $F_k(\boldsymbol{x})$ are positive definite, is a metric system but it is (in general) non-Poissonian. The answer to the question, which Poissonian system admits a global metric description, remains unknown.

Morrison [9] has pointed out that metric dynamical systems admit an asymptotic stability at isolated maxima of the function $\mathcal{S}$. To show that, let $\boldsymbol{x}$ be an isolated maximum of $\mathcal{S}$, then certainly $\partial_i \mathcal{S} = 0$ at $\boldsymbol{x}$, hence $\boldsymbol{x}$ is an equilibrium point of the semimetric dynamical system: $\dot{z}^i = \prec z^i, \mathcal{S} \succ = \sum_j G^{ij}(z) \partial_j \mathcal{S}$. Define the function $L(z) = \mathcal{S}(z) - \mathcal{S}(\boldsymbol{x})$, then obviously $L(\boldsymbol{x}) = 0$ and $L(z) < 0$ in some neighbourhood of $\boldsymbol{x}$, since $\boldsymbol{x}$ is the isolated maximum. Furthermore, $\dot{L}(z) = \prec L, \mathcal{S} \succ (z) = \prec \mathcal{S}, \mathcal{S} \succ (z) \geqslant 0$ for $z \neq \boldsymbol{x}$ and $\dot{L}(\boldsymbol{x}) = 0$, therefore L is the Lyapunov function for the system and $\boldsymbol{x}$ is its stable equilibrium point. Now, if the system is metric, then $\dot{L}(z) = \prec \mathcal{S}, \mathcal{S} \succ (z) > 0$ for $z \neq \boldsymbol{x}$ since the function $\mathcal{S}$ is not locally constant. Hence $\boldsymbol{x}$ is an asymptotically stable point. Generalization of the above construction to the infinite dimensional case is not known.

4. Constrained metric dynamics

In the framework of symplectic geometry, constrained Hamiltonian dynamics can be represented by a triplet (M, N, ω) where (M, ω) is a symplectic manifold, namely phase space, and N is a constraint submanifold of M. The antisymmetric Dirac bracket for second-class constraints [1, 16] is nothing but the Poisson bracket on some symplectic manifold $N' \subset N$, called the second-class constraint manifold [16] (also in [2]).

Similarly, constrained metric dynamics should be represented by a triplet (M, N, g) where g is a metric tensor, which is responsible for a dissipation, and N is a constraint submanifold of M. We show that the symmetric Dirac bracket for a triplet (M, N, g) is nothing but the semimetric bracket on the submanifold N. It is worthy of note that any submanifold of a Riemannian manifold is second-class with respect to the metric bracket defined by the metric tensor.

Suppose that we have a pair (M, ξ) where M is a smooth manifold and ξ is a non-degenerate symmetric $(0, 2)$-tensor on M. Then at each point $x \in M$ the map ξ_x: $T_x M \times T_x M \to R$ is bilinear, symmetric and non-degenerate and it induces a linear bijection $\xi_x^\sharp\colon T_x^*M \to T_xM$. Further, if ξ is a Riemann metric tensor, then ξ_x is a scalar product on T_xM. The non-degeneracy of the tensor ξ guarantees the existence of a (2, 0)-tensor field $\Lambda\colon T^*M \times T^*M \to R$. Let N be a submanifold of M and $\xi_{|N}$ is supposed to be non-degenerate. Then at each point x of the submanifold N, the linear space T_xM decomposes into a direct sum of a tangent space to N, T_xN, and its orthogonal with respect to bilinear symmetric functional ξ_x, i.e. $T_xM = T_xN \oplus (T_xN)^\perp$. The symmetric Dirac bracket with respect to a triplet (M, N, ξ) is defined by

$$\Lambda_{\mathrm{D}}(\alpha, \beta) = \xi(P\xi^\sharp(\alpha), P\xi^\sharp(\beta)) \quad \text{where } P \text{ is a projection onto } T_xN \text{ along } (T_xN)^\perp,$$
$$\alpha, \beta \text{ are 1-forms on } M. \tag{4.1}$$

The symmetric Dirac bracket in the space of functions then becomes

$$\prec f, g\succ_{\mathrm{D}} = \Lambda_{\mathrm{D}}(\mathrm{d}f, \mathrm{d}g) \qquad \forall f, g \in C^\infty(M). \tag{4.2}$$

If N is the second-class submanifold

$$N = \{x \in M\colon \Theta_i(x) = 0\} \tag{4.3}$$

then we derive an explicit formula for the symmetric Dirac bracket with respect to the triplet (M, N, ξ). Let X_g denote a vector field generated by the function g, i.e. $X_g(f) = \Lambda(\mathrm{d}f, \mathrm{d}g) = \prec f, g\succ = \xi(X_f, X_g)$. Let $W = [W_{ij}] = [\prec\Theta_i, \Theta_j\succ] = [\xi(X_{\Theta_i}, X_{\Theta_j})]$ and $C = [C_{ij}] = W^{-1}$. It is easy to see that the vector fields X_{Θ_i} span $TN^\perp$, then orthogonal projection Q onto $TN^\perp$ along TN has the form

$$QX = \sum_{i,j} \xi(X, X_{\Theta_i}) C_{ij} X_{\Theta_j}. \tag{4.4}$$

Then the orthogonal projection P onto TN along $TN^\perp$ is of the form $PX = X - QX$, hence we have $PX_f = X_f - \sum_{i,j} \xi(X, X_{\Theta_i}) C_{ij} X_{\Theta_j}$. Then the symmetric Dirac formula is of the form

$$\begin{aligned} \prec f, g\succ_{\mathrm{D}} &= \Lambda_{\mathrm{D}}(\mathrm{d}f, \mathrm{d}g) = \xi(PX_f, PX_g) = \xi(X_f, PX_g) \\ &= \xi(X_f, X_g) - \xi(X_f, QX_g) \\ &= \prec f, g\succ - \sum_{i,j} \prec f, \Theta_i\succ C_{ij} \prec\Theta_j, g\succ \end{aligned} \tag{4.5}$$

which coincides with the antisymmetric Dirac bracket formula (2.4) for Poisson bracket with the antisymmetric brackets $\{\cdot, \cdot\}$ replaced by $\prec\cdot, \cdot\succ$. Our procedure shown above applies to both the cases, for symmetric or antisymmetric (0, 2)-tensors.

The Dirac formula (4.5) plays a key role in the practical use of the Dirac brackets. Algebraically, one may use it as a definition of the Dirac bracket for an arbitrary symmetric or antisymmetric algebra. The disadvantage of the algebraic approach is based on the fact that it is very difficult to understand why the Jacobi identity for the new Dirac bracket holds when the above procedure is applied to a Poisson bracket. In the metric context, if the algebraic formula (4.5) is regarded as a definition of the Dirac bracket, then it is easy to check that the new algebraic Dirac bracket is a symmetric Leibniz bracket (i.e. the algebraic properties (i), (ii) hold), but the crucial non-negativity property (iii) is not easy to verify. A simple proof will be given later after theorem 4.1. It is easy to see that

$$\prec\Theta_a, f\succ_{\mathrm{D}} = 0 \qquad \text{for arbitrary function } f(z) \tag{4.6}$$

therefore all constraints Θ_a are Casimirs, i.e. belong to the centrum of the semimetric algebra $(C^\infty(X), \prec\cdot, \cdot\succ_{\mathrm{D}})$.

Now we would like to present the new formula for calculating the symmetric or antisymmetric Dirac bracket.

Theorem 4.1. [14]
The following identity holds:

$$\prec f, g\succ_{\mathrm{D}} = \frac{\det W_{f,g}}{\det W} \qquad \forall f, g \in \mathcal{F} \tag{4.7}$$

where

$$W = \begin{bmatrix} \prec\Theta_1, \Theta_1\succ & \dots & \prec\Theta_1, \Theta_N\succ \\ \prec\Theta_2, \Theta_1\succ & \dots & \prec\Theta_2, \Theta_N\succ \\ \dots & \dots & \dots \\ \prec\Theta_N, \Theta_1\succ & \dots & \prec\Theta_N, \Theta_N\succ \end{bmatrix}$$
$$W_{f,g} = \begin{bmatrix} \prec\Theta_1, \Theta_1\succ & \dots & \prec\Theta_1, \Theta_N\succ & \prec\Theta_1, g\succ \\ \prec\Theta_2, \Theta_1\succ & \dots & \prec\Theta_2, \Theta_N\succ & \prec\Theta_2, g\succ \\ \dots & \dots & \dots & \dots \\ \prec\Theta_N, \Theta_1\succ & \dots & \prec\Theta_N, \Theta_N\succ & \prec\Theta_N, g\succ \\ \prec f, \Theta_1\succ & \dots & \prec f, \Theta_N\succ & \prec f, g\succ \end{bmatrix}. \tag{4.8}$$

The same formula holds for antisymmetric Dirac brackets.

Proof. The proof is straightforward; one applies the Laplace recursive formula for the determinant expansion twice to the last column and row of the matrix $W_{f,g}$. □

One consequence of (4.7), (4.8) for a semimetric bracket is that $\forall f$ we have $\prec f, f\succ_{\mathrm{D}} = \frac{\det W_{f,f}}{\det W}$, and therefore the inequality $\prec f, f\succ_{\mathrm{D}} \geqslant 0$ holds, according to proposition 3.1. In the one constraint case, this is equivalent to the Schwartz inequality.

Theorem 4.1 provides a new effective formula for calculating Dirac brackets for both symmetric and antisymmetric cases. Usually, the direct attempt to use formula (4.5) is impractical for a system with a rather large number of constraints of second type. This is because it requires quite complicated evaluation of the elements of the inverse matrix C_{ij}. Note, however, that when we are not interested in the Dirac brackets but only in the resulting equations for constrained dynamics then the evaluation of $C = W^{-1}$ is unnecessary. Applying theorem 4.1 we immediately find the equation for the quantity f in constrained symmetric/antisymmetric dynamics:

$$\dot{f} = \prec f, \mathcal{H}\succ_{\mathrm{D}} = \frac{\det\ W_{f,\mathcal{H}}}{\det\ W} \qquad \text{or} \qquad \dot{f} = \{f, \mathcal{H}\}_{\mathrm{D}} = \frac{\det\ W_{f,\mathcal{H}}}{\det\ W}. \tag{4.9}$$

This formula is particularly convenient for finding constants of motion for constrained—symmetric and antisymmetric—dynamics. Indeed, f is a constant of motion for constrained dynamics iff $\det W_{f,\mathcal{H}} = 0$.

Below we show a few examples of the constrained symmetric Dirac brackets which are applicable in differential geometry and physics.

Example 4.1. Consider the standard Euclidean metrics $\prec z^i, z^j\succ = \delta^{ij} = G^{ij}$ and the fixed surface $f(z) = 0$ in $\mathbf{R}^n$. Here the function f is assumed to be smooth and with zero as its regular value, that is $f^{-1}(0)$ is a close regular $(n-1)$-dimensional differential submanifold in $\mathbf{R}^n$. Using $f(z) = 0$ as a constraint we find the Dirac semimetric brackets

$$\prec z^i, z^j\succ_{\mathrm{D}} = \delta^{ij} - n^i n^j \tag{4.10}$$

where $\boldsymbol{n}(z) = \frac{\nabla f}{||\nabla f||}$ is a unit normal vector to the surface at z. The metric tensor $G^{ij}_{\mathrm{D}}(z) = \delta^{ij} - n^i n^j$ is nothing but the induced metric tensor on this surface.

The following example illustrates how to derive the metric structure for lattice spins [8], that is a set of classical spins $\vec{S}_a$ where a labels the lattice sites.

Example 4.2. We introduce the usual lattice-spin metric brackets as

$$\prec S_a^i, S_b^j \succ = \delta_{ab}\delta^{ij}|S_a| = G_{ab}^{ij} \qquad i, j = 1, 2, 3 \quad a, b = 1, 2, \ldots, N \tag{4.11}$$

and we define $2N$-dimensional surface $\mathcal{P}$ by the following system of N constraints:

$$\Theta_a(\mathbf{S}) = |S_a|^2 - r_a^2 = \sum_{i=1}^{3} \left(S_a^i\right)^2 - r_a^2 = 0 \qquad a = 1, 2, \ldots, N. \tag{4.12}$$

The Dirac metric brackets for the surface $\mathcal{P}$ are

$$\prec S_a^i, S_b^j \succ_{\mathrm{D}} \; = \delta^{ij}|S_a| \left[\delta_{ab} - \frac{S_a^i S_b^j}{S_a^2}\right]. \tag{4.13}$$

□

The next example illustrates the metric structure for an energy-conserving rigid body.

Example 4.3.
One may postulate the metric brackets for a rigid body as

$$\prec \omega_i, \omega_j \succ \; = \; \delta_{ij} K(\omega) \quad \text{where } K \text{ is some function of the body angular frequency } \omega. \tag{4.14}$$

(a) Consider energy as constrained surface

$$\Theta(\omega) = \sum_{k=1}^{3} I_k \omega_k^2 - E. \tag{4.15}$$

Calculating the Dirac metric brackets one gets

$$\prec \omega_i, \omega_j \succ_{\{\Theta\}} \; = \; K(\omega) \left[\delta_{ij} - \frac{I_i I_j \omega_i \omega_j}{\sum_k I_k^2 \omega_k^2}\right]. \tag{4.16}$$

We can consider some particular cases:

(a1) One may choose $K = \sum_{k=1}^{3} I_k^2 \omega_k^2$, where I_k are moments of inertia with respect to the main axes of the rigid body. Then the Dirac metric brackets are

$$\prec \omega_i, \omega_j \succ_{\{\Theta\}} \; = \; \delta_{ij} \left[\sum_{k=1}^{3} I_k^2 \omega_k^2\right] - I_i I_j \omega_i \omega_j. \tag{4.17}$$

This metric structure coincides with the metric structure postulated by Morrison [9].

(a2) Since the Poisson structure of a rigid body is the same as for classical spins, we may postulate $K = |\omega|$, where $|\omega| = \sqrt{\sum_{k=1}^{3} \omega_k^2}$. The Dirac metric brackets follow

$$\prec \omega_i, \omega_j \succ_{\{\Theta\}} \; = \; |\omega| \left[\delta_{ij} - \frac{I_i I_j \omega_i \omega_j}{\sum_{k=1}^{3} I_k^2 \omega_k^2}\right]. \tag{4.18}$$

(b) Consider the Poissonian Casimir as a constrained surface

$$\Theta(\omega) = \sum_{k=1}^{3} \omega_k^2 - |\omega_0|^2. \tag{4.19}$$

The Dirac metric brackets follow

$$\prec \omega_i, \omega_j \succ_{\{\Theta\}} \; = \; K(\omega) \left[\delta_{ij} - \frac{\omega_i \omega_j}{|\omega|^2}\right]. \tag{4.20}$$

For instance, $K(\omega) = |\omega|$, we get

$$\prec\omega_i, \omega_j\succ_{\{\Theta\}} = |\omega| \left[\delta_{ij} - \frac{\omega_i\omega_j}{|\omega|^2}\right]. \tag{4.21}$$

The next examples show how our formalism works in the Hilbert spaces.

Example 4.4. Let $Ph = L^2(\mathbf{R}^n; \mathbf{R}^d)$ be a Hilbert space of real, vector-valued square integrable functions with standard scalar product $\langle\cdot|\cdot\rangle$, i.e. $\langle \boldsymbol{f}|\boldsymbol{g}\rangle = \int \mathrm{d}^n\boldsymbol{x}\, \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x})$, where $\boldsymbol{f} = (f_1, \ldots, f_d), \boldsymbol{g} = (g_1, \ldots, g_d)$. Let $||\boldsymbol{f}||^2 = \langle \boldsymbol{f}|\boldsymbol{f}\rangle$. In the space of all smooth functionals over *Ph*, according to the construction given in example 3.2, the semimetric structure may be defined by

$$\prec\phi_1, \phi_2\succ(\boldsymbol{f}) = \sum_{i=1}^{d} \int \mathrm{d}^n\boldsymbol{x} \left[\frac{\delta\phi_1}{\delta f_i(\boldsymbol{x})}\frac{\delta\phi_2}{\delta f_i(\boldsymbol{x})}\right] \qquad \phi_1, \phi_2\colon Ph \to R \tag{4.22}$$

where $\frac{\delta\phi_i}{\delta f_k}$ denotes a Gateaux functional derivative.

Consider a surface of infinite dimensional sphere S^∞ with radius r as a subspace with one constraint

$$S^\infty = \{\boldsymbol{f} \in Ph\colon ||\boldsymbol{f}||^2 = r^2\}. \tag{4.23}$$

Now we calculate the Dirac metric structure for the sphere S^∞. The metric bracket (4.22) can be rewritten; introduce the *canonical metric tensor G*

$$G^{ij}(\boldsymbol{x}, \boldsymbol{y}) = \prec f_i(\boldsymbol{x}), f_j(\boldsymbol{y})\succ = \delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{4.24}$$

as

$$\prec\phi_1, \phi_2\succ(\boldsymbol{f}) = \int \mathrm{d}^n\boldsymbol{x}\mathrm{d}^n\boldsymbol{y}\frac{1}{2}\left[\frac{\delta\phi_1}{\delta f_i(\boldsymbol{x})}\frac{\delta\phi_2}{\delta f_j(\boldsymbol{y})} + \frac{\delta\phi_1}{\delta f_j(\boldsymbol{y})}\frac{\delta\phi_2}{\delta f_i(\boldsymbol{x})}\right]\prec f_i(\boldsymbol{x}), f_j(\boldsymbol{y})\succ. \tag{4.25}$$

The Dirac semimetric structure on S^∞ follows

$$\prec f_i(\boldsymbol{x}), f_j(\boldsymbol{y})\succ_{\mathrm{D}} = \delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}) - \frac{f_i(\boldsymbol{x})f_j(\boldsymbol{y})}{||\boldsymbol{f}||^2}. \tag{4.26}$$

Example 4.5. Let $\Psi = \Psi_1 + \mathrm{i}\Psi_2$ and its complex conjugate $\Psi^* = \Psi_1 - \mathrm{i}\Psi_2$ where Ψ_1, Ψ_2 are real functions integrated by square, i.e. they belong to the Hilbert space L^2. We define metric structure by

$$\prec\Psi_k(\boldsymbol{x}), \Psi_l(\boldsymbol{y})\succ = \tfrac{1}{2}\delta_{kl}\delta(\boldsymbol{x} - \boldsymbol{y}) \qquad \text{where} \quad k, l = 1, 2. \tag{4.27}$$

One can rewrite it in the form

$$\prec\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\succ = \prec\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ = 0 \qquad \prec\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{4.28}$$

which we call *canonical metric bracket for quantum mechanics*. The Dirac structure on the sphere $||\Psi|| = const$ in the Hilbert space follows

$$\begin{aligned} &\prec\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\succ_{\mathrm{D}} = -\frac{\Psi(\boldsymbol{x})\Psi(\boldsymbol{y})}{2||\Psi||^2} \qquad \prec\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ_{\mathrm{D}} = -\frac{\Psi^*(\boldsymbol{x})\Psi^*(\boldsymbol{y})}{2||\Psi||^2} \\ &\prec\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ_{\mathrm{D}} = \delta(\boldsymbol{x} - \boldsymbol{y}) - \frac{\Psi(\boldsymbol{x})\Psi^*(\boldsymbol{y})}{2||\Psi||^2}. \end{aligned} \tag{4.29}$$

One gets the Dirac brackets of the components Ψ_k in accordance with (4.26)

$$\prec\Psi_k(\boldsymbol{x}), \Psi_l(\boldsymbol{y})\succ_{\mathrm{D}} = \frac{1}{2}\left[\delta_{kl}\delta(\boldsymbol{x} - \boldsymbol{y}) - \frac{\Psi_k(\boldsymbol{x})\Psi_l(\boldsymbol{y})}{||\Psi||^2}\right]. \tag{4.30}$$

Example 4.6. (The canonical description for incompressible, viscous fluid dynamics is based on this example). Let $Ph = W^{(1,2)}(\mathbf{R}^n; \mathbf{R}^n)$ be a Sobolev space of real functions. In the space of all smooth functionals over Ph we introduce a semimetric structure

$$\prec\phi_1, \phi_2\succ(\boldsymbol{J}) = \int \mathrm{d}^n\boldsymbol{x} \left\{ a\left[\nabla \cdot \frac{\delta\phi_1}{\delta\boldsymbol{J}(\boldsymbol{x})}\right]\left[\nabla \cdot \frac{\delta\phi_2}{\delta\boldsymbol{J}(\boldsymbol{x})}\right] + b\sum_i \nabla\left[\frac{\delta\phi_1}{\delta J_i(\boldsymbol{x})}\right] \cdot \nabla\left[\frac{\delta\phi_2}{\delta J_i(\boldsymbol{x})}\right]\right\} \tag{4.31}$$

where $\boldsymbol{J}$ denotes the real vector in the n-dimensional Euclidean space and a, b are real non-negative coefficients. We can rewrite this semimetric structure in the form

$$\prec J_i(\boldsymbol{x}), J_j(\boldsymbol{y})\succ = -\left[a\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + b\delta^{ij}\triangle\right]\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{4.32}$$

Consider infinite-dimensional subspace $\mathcal{V}$ of divergence-free functions (incompressibility condition) as a system with an infinite number of constraints:

$$\mathcal{V} = \{\boldsymbol{J} \in Ph: \forall \boldsymbol{x}\ \Theta_{\boldsymbol{x}}(\boldsymbol{J}) = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}(\boldsymbol{x}) = 0\}. \tag{4.33}$$

Then the Dirac semimetric structure for the subspace $\mathcal{V}$ follows:

$$\begin{aligned}\prec J_i(\boldsymbol{x}), J_j(\boldsymbol{y})\succ_{\mathrm{D}} &= -\left[a\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + b\delta^{ij}\triangle\right]\delta(\boldsymbol{x}-\boldsymbol{y}) \\ &\quad - \int \mathrm{d}z\, \mathrm{d}z' \prec J_i(\boldsymbol{x}), \Theta(z)\succ C(z, z')\prec\Theta(z'), J_j(\boldsymbol{y})\succ\end{aligned} \tag{4.34}$$

where C is an inverse symmetric operator of the constraint matrix

$$C(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{a+b}\int \mathrm{d}z\, G(|\boldsymbol{x}-z|)G(|z-\boldsymbol{y}|) \tag{4.35}$$

here $G(|\boldsymbol{x}-\boldsymbol{y}|)$ denotes the standard Green function (fundamental distribution) of the Laplace equation, i.e. $\triangle_{\boldsymbol{x}} G(|\boldsymbol{x}-\boldsymbol{y}|) = \delta(\boldsymbol{x}-\boldsymbol{y})$. Putting equation (4.35) back into equation (4.34) and after some simple calculations finally we obtain

$$\prec J_i(\boldsymbol{x}), J_j(\boldsymbol{y})\succ_{\mathrm{D}} = -b\left[\delta_{ij}\triangle_{\boldsymbol{x}} - \frac{\partial^2}{\partial x_i \partial x_j}\right]\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{4.36}$$

Physically, the above equation (4.36) fully describes *dissipative structure for an incompressible viscous fluid*. We shall see that more clearly in section 6.5.

5. Semimetric–Poissonian systems

Physical systems are usually dissipative. It turns out that both Poissonian and metric structures *alone* are not enough to describe dissipative systems. However, a proper combination of these two types of dynamics can, for many interesting cases, provide a satisfactory and fully algebraic description of dissipative dynamics.

A *semimetric–Poissonian bracket* on the space of functions $\mathcal{F} = C^\infty(X)$ is a bilinear operation $\{\{\cdot, \cdot\}\}: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ which is a linear combination of a Poisson and a semimetric bracket

$$\forall f, g \in \mathcal{F}: \{\{f, g\}\} = \{f, g\} - \prec f, g\succ \tag{5.1}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket and $\prec\cdot, \cdot\succ$ is the semimetric bracket.

Definition 5.1. *The semimetric–Poisson manifold is a pair* $(M, \Pi - G)$ *where* Π *is a Poisson tensor,* G *is a semimetric tensor.*

We shall call a *semimetric–Poissonian dynamics* a dynamics governed by the following system of equations:

$$\dot{z}^i = \{\{z^i, \Phi\}\} = \{z^i, \Phi\} - \prec z^i, \Phi\succ = X_\Phi(z^i) - X^{\mathrm{D}}_\Phi(z^i) \tag{5.2}$$

where Φ is some phase space function. In real physical applications it is often the case that Φ has the interpretation of the system free energy. It is then a matter of convention to choose a minus sign in equation (5.2). Indeed, we have then

$$\dot{\Phi} = \{\Phi, \Phi\} - \prec\Phi, \Phi\succ = -\prec\Phi, \Phi\succ \leqslant 0 \tag{5.3}$$

what describes the dissipation of energy.

It is often even more convenient to go a step further and decompose function Φ into two parts: the internal energy $\mathcal{H}$ and the dissipation function $\mathcal{S}$, so $\Phi = \mathcal{H} - \mathcal{S}$. Hence, when $\mathcal{S}$ is a Casimir of the Poissonian part, $\{f, \mathcal{S}\} = 0$ for all f, the evolution of some 'observable' f in the semimetric–Poissonian dynamics follows

$$\dot{f} = \{\{f, \Phi\}\} = \{f, \mathcal{H}\} - \prec f, \mathcal{H} - \mathcal{S}\succ. \tag{5.4}$$

Equation (5.4) for $f = \mathcal{S}$ gives us the semimetric–Poissonian formulation for the 'second law of thermodynamics', namely

$$\dot{\mathcal{S}} + \prec\mathcal{S}, \mathcal{H}\succ \; = \; \prec\mathcal{S}, \mathcal{S}\succ \geqslant 0 \tag{5.5}$$

where the expression on the left-hand side of equation (5.5) is just the convective time derivative of $\mathcal{S}$ along the time trajectory in the semimetric–Poissonian phase space.

Note that if $\boldsymbol{x}$ is an isolated minimum of Φ, then the function $L(z) = \Phi(z) - \Phi(\boldsymbol{x})$ is a Lyapunov function for the semimetric-Poisson system. Hence, we obtain

Proposition 5.1. *If $\boldsymbol{x}$ is an isolated minimum of the free energy function Φ, then $\boldsymbol{x}$ is a stable equilibrium point for the semimetric–Poissonian system $\dot{z} = \{z, \Phi\} - \prec z, \Phi\succ$. Furthermore, if the system is metric–Poissonian, then $\boldsymbol{x}$ is an asymptotically stable point.*

Example 5.1. Consider a modification of the harmonic oscillator described by

$$\dot{x}_1 = x_2 - ax_1\left(x_1^2 + x_2^2\right) \qquad \dot{x}_2 = -x_1 - ax_2\left(x_1^2 + x_2^2\right). \tag{5.6}$$

This system is semimetric–Poissonian with

$$\begin{aligned} &\{x_1, x_2\} = 1 \qquad \Phi = \tfrac{1}{2}\left[x_1^2 + x_2^2\right] \\ &\prec x_1, x_1\succ = ax_1^2 \qquad \prec x_2, x_2\succ = ax_2^2 \qquad \prec x_1, x_2\succ = ax_1x_2. \end{aligned} \tag{5.7}$$

Point (0, 0) is an isolated minimum of Φ, hence it is a stable equilibrium point for the system. □

The proposed formal structure of Poissonian and semimetric dynamics in the preceding sections allows us to suggest the following scheme for the construction of constrained dissipative dynamics of real physical systems:

(a) Consider a canonical Poisson structure and a semimetric structure. The semimetric structure must be postulated according to our physical insight into the nature of relevant dissipative processes.
(b) Choose the sets A and B of the second-class constraints for the Poissonian and semimetric part, respectively. Note that any constraint is second-class for non-degenerate semimetric structure.
(c) Calculate antisymmetric and symmetric Dirac brackets with respect to the set of the constraints A and B and later take a union of them.

The most interesting cases take place when the set of constraints for the semimetric part is a subset of constants of motion for the Poissonian part, i.e. the system is dissipative; however, some of the constants of the Poisson dynamics remain constants for the semimetric–Poissonian dynamics. One may use this feature to design many interesting dissipative systems. As an example, we illustrate how to design a variety of damped rigid body dynamics in the next section.

6. Applications: physical examples

6.1. Particle on the hypersurface with friction

We shall begin our illustration of the Dirac bracket applications by discussion of a simple example, namely the classical particle moving with friction on a hypersurface $\mathcal{S}_{\text{config}} = \{\boldsymbol{x} \in \mathbf{R}^n \,|\, f(\boldsymbol{x}) = 0\}$. Denote the particle position by $\boldsymbol{x}$ and its conjugate momentum by $\boldsymbol{p}$ and let the friction force experienced by that particle be proportional to the particle velocity. The phase space $\mathbf{R}^{2n}$ is now equipped with two structures: canonical Poisson and semimetric structure. Then the following Poisson structure is described by

$$\{x_i, x_j\} = 0 = \{p_i, p_j\} \quad \{x_i, p_j\} = \delta_{ij} \tag{6.1}$$

and the semimetric structure is defined by

$$\prec x_i, x_j \succ = 0 = \prec x_i, p_j \succ \qquad \prec p_i, p_j \succ = \delta_{ij}\lambda_i \tag{6.2}$$

where $\lambda_i(\boldsymbol{x}) > 0$ is the directional and space-dependent damping coefficient.

The semimetri–Poisson structure is defined by $\{\{\cdot,\cdot\}\} = \{\cdot,\cdot\} - \prec\cdot,\cdot\succ$.

Consider $\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})$, the dissipative dynamics derived from the above structures follows

$$\dot{x}_i = \{\{x_i, \mathcal{H}\}\} = \frac{p_i}{m} \quad \dot{p}_i = \{\{p_i, \mathcal{H}\}\} = -\lambda_i(\boldsymbol{x})\frac{p_i}{m} - \frac{\partial V}{\partial x_i} \tag{6.3}$$

which can be rewritten in the Newtonian form $m\ddot{x}_i + \lambda_i(\boldsymbol{x})\dot{x}_i - F_i(\boldsymbol{x}) = 0$, where $\boldsymbol{F}(\boldsymbol{x}) = -\frac{\partial V}{\partial \boldsymbol{x}}$ is a potential force. In particular, when $\lambda_i(\boldsymbol{x}) = \lambda(\boldsymbol{x})$, we have $m\ddot{\boldsymbol{x}} + \lambda(\boldsymbol{x})\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}) = 0$.

Next consider a fixed algebraic surface $f(\boldsymbol{x}) = 0$ in $\mathbf{R}^n$. We make the assumptions that f is smooth and zero is its regular value. The second assumption ensures that $\mathcal{S}_{\text{config}} = f^{-1}(0)$ is a close regular $(n-1)$-dimensional differential submanifold in $\mathbf{R}^n$. Moreover, this assumption guarantees $\nabla f \neq 0$; so we can use the gradient to define the normal vector on $\mathcal{S}_{\text{config}}$.

The set of constraints now consists of two elements:

$$\Theta_1 \equiv f(\boldsymbol{x}) = 0 \quad \Theta_2 \equiv \boldsymbol{p} \cdot \frac{\partial f}{\partial \boldsymbol{x}} = 0. \tag{6.4}$$

For the Poissonian dynamics both the above constraints Θ_i are second-class in the Dirac classification.

Denoting the unit normal vector to the surface f at the point x by $\boldsymbol{n}(x)$, $\boldsymbol{n}(\boldsymbol{x}) = \frac{1}{|\partial f/\partial \boldsymbol{x}|}\frac{\partial f}{\partial \boldsymbol{x}}$, the antisymmetric Dirac brackets for the Poissonian part of our construction are

$$\{x_i, x_j\}_{\mathrm{D}} = 0 \qquad \{x_i, p_j\}_{\mathrm{D}} = \delta_{ij} - \frac{1}{\left|\frac{\partial f}{\partial \boldsymbol{x}}\right|^2}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = \delta_{ij} - n_i(\boldsymbol{x})n_j(\boldsymbol{x})$$

$$\begin{aligned}\{p_i, p_j\}_{\mathrm{D}} &= \frac{1}{\left|\frac{\partial f}{\partial \boldsymbol{x}}\right|^2}\left\{\frac{\partial f}{\partial x_j}\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_i}\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]\frac{\partial f}{\partial x_j}\right\} \\ &= n_j(\boldsymbol{x})\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]n_i(\boldsymbol{x}) - n_i(\boldsymbol{x})\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]n_j(\boldsymbol{x}).\end{aligned} \tag{6.5}$$

For the semimetric dynamics only the second constraint Θ_2 is second-class. The symmetric Dirac brackets for the semimetric part

$$\prec x_i, x_j \succ_{\mathrm{D}} = 0 = \prec x_i, p_j \succ_{\mathrm{D}} \qquad \prec p_i, p_j \succ_{\mathrm{D}} = \delta_{ij}\lambda_i - \frac{\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{x})\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}}{\sum_k \lambda_k(\boldsymbol{x})\left|\frac{\partial f}{\partial x_k}\right|^2}. \tag{6.6}$$

In particular, when $\lambda_i(\boldsymbol{x}) = \lambda(\boldsymbol{x})$, the symmetric Dirac brackets can be written in the form

$$\prec x_i, x_j \succ_{\mathrm{D}} = 0 = \prec x_i, p_j \succ_{\mathrm{D}} \qquad \prec p_i, p_j \succ_{\mathrm{D}} = \lambda(\boldsymbol{x})[\delta_{ij} - n_i(\boldsymbol{x})n_j(\boldsymbol{x})]. \tag{6.7}$$

Finally take a union of these two Dirac structures

$$\{\{x_i, x_j\}\}_{\mathrm{D}} = 0 \qquad \{\{x_i, p_j\}\}_{\mathrm{D}} = \delta_{ij} - n_i(\boldsymbol{x})n_j(\boldsymbol{x})$$

$$\{\{p_i, p_j\}\}_{\mathrm{D}} = \frac{1}{\left|\frac{\partial f}{\partial \boldsymbol{x}}\right|^2}\left\{\frac{\partial f}{\partial x_j}\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_i}\left[\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right]\frac{\partial f}{\partial x_j}\right\}$$
$$- \lambda_i(\boldsymbol{x})\left[\delta_{ij} - \frac{\lambda_j(\boldsymbol{x})\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}}{\sum_k \lambda_k(\boldsymbol{x})\left|\frac{\partial f}{\partial x_k}\right|^2}\right]. \tag{6.8}$$

When the Hamiltonian for that system has the form $\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})$, the dissipative Hamilton–Dirac equations of motion follow as

$$\dot{x}_i = \{x_i, \mathcal{H}\} - \prec x_i, \mathcal{H}\succ = \frac{1}{m}[p_i - (\boldsymbol{p}\cdot\boldsymbol{n})n_i] = \frac{p_i}{m}$$

$$\dot{p}_i = \{p_i, \mathcal{H}\} - \prec p_i, \mathcal{H}\succ = F_i - \left[\boldsymbol{F}\cdot\boldsymbol{n} + \frac{1}{m}\boldsymbol{p}\cdot\left[\left(\boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{n}\right]\right]n_i$$
$$- \frac{\lambda_i(\boldsymbol{x})}{m}\left[p_i - \frac{\partial f}{\partial x_i}\left(\frac{\sum_j \lambda_j(\boldsymbol{x})\frac{\partial f}{\partial x_j}p_j}{\sum_k \lambda_k(\boldsymbol{x})\left|\frac{\partial f}{\partial x_k}\right|^2}\right)\right]. \tag{6.9}$$

For isotropic damping, when $\lambda_i(\boldsymbol{x}) = \lambda(\boldsymbol{x})$, we can rewrite these equations of motion in the Newtonian form as

$$m\ddot{\boldsymbol{x}} + \lambda(\boldsymbol{x})\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) - \left[\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) + m\dot{\boldsymbol{x}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{n}(\boldsymbol{x})\right]\boldsymbol{n}(\boldsymbol{x}). \tag{6.10}$$

6.2. Variety dynamics of damped rigid body

The usual Poisson brackets for a rigid body angular velocity vector $\vec{\omega}$, after suitable rescaling, are

$$\{\omega_i, \omega_j\} = \varepsilon_{ijk}\omega_k. \tag{6.11}$$

Suppose that there is no second-type constraint for the Poisson part of the body dynamics and just one such a constraint for its symmetric part. Assume now that this constraint is such that the system energy (or any Casimir function for the metric part) is constant. The choice of that constraint determines the details of the semimetric–Poissonian structure.

Using the canonical metric brackets (4.16) and the constrained energy $\mathcal{H}$, from example 4.3 we obtain the Dirac metric bracket for a rigid body (4.17). Combining these two structures one finds the metric–Poissonian brackets for a rigid body [9]:

$$\{\{\omega_i, \omega_j\}\} = \{\omega_i, \omega_j\} - \lambda\prec\omega_i, \omega_j\succ_{\mathrm{D}} = \varepsilon_{ijk}\omega_k - \lambda\left[\delta_{ij}\left(\sum_{k=1}^{3} I_k^2\omega_k^2\right) - I_i I_j\omega_i\omega_j\right]. \tag{6.12}$$

Consider the system 'free energy'

$$\Phi = \mathcal{H} - \mathcal{S} = \frac{1}{2}\left[\sum_{k=1}^{3} I_k \omega_k^2\right] - \mathcal{S}(|\omega|^2) \tag{6.13}$$

The equations of energy-conserving motion for the damped body are

$$\begin{aligned}\dot{\omega}_1 &= \{\{\omega_1, \Phi\}\} = \{\omega_1, \mathcal{H}\} + \lambda \prec \omega_1, \mathcal{S} \succ_{\mathrm{D}} \\ &= \omega_2\omega_3(I_2 - I_3) + 2\lambda\mathcal{S}'\omega_1\left[I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2\right] \\ &\quad \text{and its cyclic permutation.}\end{aligned} \tag{6.14}$$

The dissipation of the free energy follows

$$\dot{\Phi} = -\dot{\mathcal{S}} = -\lambda \prec \mathcal{S}, \mathcal{S} \succ_{\mathrm{D}} = -\lambda\left[\left|\frac{\partial \mathcal{S}}{\partial \omega}\right|^2 \left(\sum_{k=1}^{3} I_k^2 \omega_k^2\right) - \left(I_i \omega_i \frac{\partial \mathcal{S}}{\partial \omega_i}\right)^2\right] \leqslant 0 \tag{6.15}$$

and there is no dissipation of energy iff $I_1 = I_2 = I_3$.

Alternatively, we can derive a new semimetric–Poissonian bracket by combining metric bracket (4.18) with the standard Poisson bracket for the rigid body

$$\{\{\omega_i, \omega_j\}\} = \{\omega_i, \omega_j\} - \lambda \prec \omega_i, \omega_j \succ_{\mathrm{D}} = \varepsilon_{ijk}\omega_k - \lambda|\omega|\left[\delta_{ij} - \frac{I_i I_j \omega_i \omega_j}{\sum_{k=1}^{3} I_k^2 \omega_k^2}\right]. \tag{6.16}$$

Equations of motion following from the free energy (6.13) are

$$\begin{aligned}\dot{\omega}_1 &= \{\{\omega_1, \Phi\}\} = \{\omega_1, \mathcal{H}\} + \lambda \prec \omega_1, \mathcal{S} \succ_{\mathrm{D}} \\ &= \omega_2\omega_3(I_2 - I_3) + 2\lambda\mathcal{S}'|\omega|\omega_1 \frac{I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2}{\sum_{k=1}^{3} I_k^2 \omega_k^2} \\ &\quad \text{and its cyclic permutation.}\end{aligned} \tag{6.17}$$

Finally, we derive a new semimetric–Poissonian bracket by combining metric bracket (4.21) with the standard Poisson bracket for the rigid body

$$\{\{\omega_i, \omega_j\}\} = \{\omega_i, \omega_j\} - \lambda \prec \omega_i, \omega_j \succ_{\mathrm{D}} = \varepsilon_{ijk}\omega_k - \lambda|\omega|\left[\delta_{ij} - \frac{\omega_i \omega_j}{|\omega|^2}\right]. \tag{6.18}$$

Again using the free energy (6.13) we find

$$\begin{aligned}\dot{\omega}_1 &= \{\{\omega_1, \Phi\}\} = \{\omega_1, \mathcal{H}\} - \lambda \prec \omega_1, \mathcal{H} \succ_{\mathrm{D}} \\ &= \omega_2\omega_3(I_2 - I_3) - \lambda|\omega|\omega_1\left[I_1 - \frac{\sum_{k=1}^{3} I_k \omega_k^2}{|\omega|^2}\right] \\ &\quad \text{and its cyclic permutation.}\end{aligned} \tag{6.19}$$

The above dynamics does not conserve the energy, but $|\omega|^2$ remains the system Casimir. The energy dissipation is given by

$$\dot{\mathcal{H}} = -\lambda \prec \mathcal{H}, \mathcal{H} \succ_{\mathrm{D}} = -\lambda|\omega|\left[\sum_k I_k^2 \omega_k^2 - \frac{\left(\sum_k I_k \omega_k^2\right)^2}{|\omega|^2}\right] \leqslant 0 \tag{6.20}$$

since $\left(\sum_k I_k^2 \omega_k^2\right)|\omega|^2 \geqslant \left(\sum_k I_k \omega_k^2\right)^2$. For $\lambda > 0$, $|\omega| > 0$, there is no dissipation of energy when $I_1 = I_2 = I_3$.

6.3. Classical damped spins

The Poisson bracket for lattice spins

$$\left\{S_a^i, S_b^j\right\} = \delta_{ab}\varepsilon_{ijk}S_a^k \tag{6.21}$$

where a labels the spin location and $k = 1, 2, 3$, guarantees that the length of each spin $|\boldsymbol{S}_a|$ is a Casimir. Our model is based on the assumption that there are no second-class constraints for the Poissonian part and only one constraint for the metric part.

Starting from the canonical metric bracket (4.11) and using the length of the spins $|\boldsymbol{S}_a|$ constraint, as in example 4.2, we obtain the Dirac metric bracket for lattice spins (4.13). Combining these two structures one finds the metric–Poissonian of lattice spins [8]:

$$\left\{\left\{S_a^i, S_b^j\right\}\right\} = \left\{S_a^i, S_b^j\right\} - \lambda \prec S_a^i, S_b^j \succ_{\mathrm{D}} = \delta_{ab}\varepsilon_{ijk}S_a^k - \lambda\delta^{ij}|\boldsymbol{S}_a|\left[\delta_{ab} - \frac{S_a^i S_b^j}{S_a^2}\right] \tag{6.22}$$

where λ is the damping coefficient.

Now we can easily derive the Landau–Lifshitz–Gilbert equation of classical damped lattice spins

$$\dot{\boldsymbol{S}}_a = \{\{\boldsymbol{S}_a, \mathcal{H}\}\} = \boldsymbol{S}_a \times \boldsymbol{B}_{ef,a} - \lambda\frac{\boldsymbol{S}_a \times (\boldsymbol{S}_a \times \boldsymbol{B}_{ef,a})}{|\boldsymbol{S}_a|} \tag{6.23}$$

where $\boldsymbol{B}_{ef,a} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{S}_a}$ is the effective magnetic field acting on the spin $\boldsymbol{S}_a$.

The dissipation of energy follows

$$\dot{\mathcal{H}} = -\lambda \prec \mathcal{H}, \mathcal{H} \succ_{\mathrm{D}} = -\lambda\sum_a |S_a|\left[\boldsymbol{B}_{ef,a}^2 - \frac{(\boldsymbol{B}_{ef,a}\cdot\boldsymbol{S}_a)^2}{|\boldsymbol{S}_a|^2}\right] \leqslant 0. \tag{6.24}$$

6.4. Dissipative quantum mechanics

Combining canonical Poisson brackets for quantum mechanics (QM)

$$\{\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\} = \{\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\} = 0 \qquad \{\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\} = \frac{1}{\mathrm{i}\hbar}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{6.25}$$

and constrained metric brackets for QM (4.29), which were derived from the canonical metric bracket for QM (4.28) for the physically important constraint, namely the conservation of the wavefunction norm (probability conservation) as in example 4.5, we find a *dissipative metric–Poissonian structure for QM*:

$$\begin{aligned}
\{\{\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\}\} &= \{\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\} - \frac{\lambda}{\hbar}\prec\Psi(\boldsymbol{x}), \Psi(\boldsymbol{y})\succ_{\mathrm{D}} = \frac{\lambda}{\hbar}\frac{\Psi(\boldsymbol{x})\Psi(\boldsymbol{y})}{2||\Psi||^2} \\
\{\{\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\}\} &= \{\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\} - \frac{\lambda}{\hbar}\prec\Psi^*(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ_{\mathrm{D}} = \frac{\lambda}{\hbar}\frac{\Psi^*(\boldsymbol{x})\Psi^*(\boldsymbol{y})}{2||\Psi||^2} \\
\{\{\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\}\} &= \{\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\} - \frac{\lambda}{\hbar}\prec\Psi(\boldsymbol{x}), \Psi^*(\boldsymbol{y})\succ_{\mathrm{D}} \\
&= \frac{1}{\mathrm{i}\hbar}\delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{\lambda}{\hbar}\left[\delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{\Psi(\boldsymbol{x})\Psi^*(\boldsymbol{y})}{2||\Psi||^2}\right]
\end{aligned} \tag{6.26}$$

where λ is the (undefined) damping constant.

Using the conventional Hamiltonian for the Schrödinger equation

$$\mathcal{H}(\Psi, \Psi^*) = \langle\Psi|H|\Psi\rangle = \int \mathrm{d}^n\boldsymbol{x}\,\Psi^*(\boldsymbol{x})H\Psi(\boldsymbol{x}) \tag{6.27}$$

where H is the quantum mechanical (self-adjoined) operator, we obtain the evolution of the wavefunction in the form

$$\mathrm{i}\hbar\partial_t\Psi(\boldsymbol{x}) = \mathrm{i}\hbar\{\{\Psi(\boldsymbol{x}),\mathcal{H}\}\} = H\Psi(\boldsymbol{x}) + \mathrm{i}\lambda\left[\frac{\langle\Psi|H|\Psi\rangle}{||\Psi||^2} - H\right]\Psi(\boldsymbol{x}). \tag{6.28}$$

This equation is known as the *Gisin dissipative wave equation* [11]. Here the construction of the semimetric bracket ensures that the norm of the state vector is reserved (so the probability is conserved) since it is a second-class constraint for the semimetric part. The dissipation of energy follows

$$\dot{\mathcal{H}} = -\prec\mathcal{H},\mathcal{H}\succ = \frac{2\lambda}{\hbar}\left[-||H\Psi||^2 + \frac{\langle\Psi|H|\Psi\rangle^2}{||\Psi||^2}\right] \leqslant 0$$
$$\text{(Schwartz inequality).} \tag{6.29}$$

In the above equation the equality is achieved for Ψ which are the eigenstates of the Hamiltonian. The damping in the Gisin equation refers, therefore, to the transition amplitudes only. When the initial wave packet is constructed from eigenstates corresponding to energies $E \geqslant E_0$ then the final state of the evolution described by equation (6.28) is the eigenstate with energy E_0. This property distinguishes the Gisin dissipative wave equation from the other dissipative Schrödinger equations.

6.5. *Viscous fluid dynamics*

In fluid mechanics the state of an isothermal fluid is described by its mass density and velocity fields $(\varrho,\boldsymbol{u})$ or by $(\varrho,\boldsymbol{J})$ where the current $\boldsymbol{J}$ field equals $\boldsymbol{J}=\varrho\boldsymbol{u}$. The semimetric structure for fluid dynamics [8] follows

$$\begin{aligned}\prec\varrho(\boldsymbol{x}),\varrho(\boldsymbol{y})\succ &= 0 \qquad \prec\varrho(\boldsymbol{x}),J_k(\boldsymbol{y})\succ = 0\\ \prec J_k(\boldsymbol{x}),J_l(\boldsymbol{y})\succ &= -\left[a\frac{\partial^2}{\partial x_k\partial x_l} + \eta\delta_{kl}\triangle_x\right]\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned} \tag{6.30}$$

where $a=\zeta+\frac{\eta}{3}$ and ζ,η are the bulk and shear viscosity, respectively.

The structure (6.30) is semimetric, indeed $\forall\mathcal{G}$ we have

$$\begin{aligned}\prec\mathcal{G},\mathcal{G}\succ(\varrho,\boldsymbol{J}) &= -\sum_{k,l}\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\frac{\delta\mathcal{G}}{\delta J_k(\boldsymbol{x})}\left\{\left[a\frac{\partial^2}{\partial x_k\partial x_l} + \eta\delta_{kl}\triangle_x\right]\delta(\boldsymbol{x}-\boldsymbol{y})\right\}\frac{\delta\mathcal{G}}{\delta J_l(\boldsymbol{y})}\\ &= \int \mathrm{d}\boldsymbol{x}\left\{a\left[\nabla\cdot\frac{\delta\mathcal{G}}{\delta\boldsymbol{J}(\boldsymbol{x})}\right]^2 + \eta\sum_k\left|\nabla\left[\frac{\delta\mathcal{G}}{\delta J_k(\boldsymbol{x})}\right]\right|^2\right\}.\end{aligned} \tag{6.31}$$

Note that this is not a metric bracket, since any functional which depends only on ϱ has zero semimetric bracket with itself. Also note that the kinetic fluid energy $E_{\mathrm{kin}} = \int \mathrm{d}\boldsymbol{x}\,\boldsymbol{J}^2/2\varrho$ dissipation follows directly from (6.30)

$$\dot{E}_{\mathrm{kin}} = -\prec E_{\mathrm{kin}},E_{\mathrm{kin}}\succ = -\int \mathrm{d}\boldsymbol{x}\left\{a\left[\nabla\cdot\frac{\boldsymbol{J}}{\varrho(\boldsymbol{x})}\right]^2 + \eta\sum_k\left|\nabla\frac{J_k}{\varrho(\boldsymbol{x})}\right|^2\right\} \leqslant 0. \tag{6.32}$$

In classical hydrodynamics a particular role is played by the incompressible fluid assumption, for the incompressible viscous fluid according to equation (4.36) we have

$$\begin{aligned}\prec\varrho(\boldsymbol{x}),\varrho(\boldsymbol{y})\succ_{\mathrm{D}} &= 0 \quad \prec\varrho(\boldsymbol{x}),J_i(\boldsymbol{y})\succ_{\mathrm{D}} = 0\\ \prec J_i(\boldsymbol{x}),J_j(\boldsymbol{y})\succ_{\mathrm{D}} &= -\eta\left[\delta_{ij}\triangle_x - \frac{\partial^2}{\partial x_i\partial x_j}\right]\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{6.33}$$

The dissipative energy for the viscous incompressible fluid should be easily calculated as follows:

$$\dot{E}_{\mathrm{kin}} = -\prec E_{\mathrm{kin}}, E_{\mathrm{kin}} \succ_{\mathrm{D}} = -\frac{\eta}{2} \int \mathrm{d}\boldsymbol{x} \left\{ \sum_{k,l} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)^2 \right\} \leqslant 0. \tag{6.34}$$

Finally, put $\nu = \frac{\eta}{\varrho_0}$, the kinematic viscosity, using (6.33) and from the Poisson part calculated in [7] we easily derive non-local evolutional equations for the incompressible viscous fluid

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \triangle \boldsymbol{u} = \nabla \int \mathrm{d}\boldsymbol{z}\, G(\boldsymbol{x} - \boldsymbol{z}) \nabla_z \left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \triangle_z \boldsymbol{u} \right]. \tag{6.35}$$

7. Final comments

The description of the dissipative systems dynamics is usually undertaken within the framework of non-equilibrium statistical mechanics. The ‘simplified’ version of the full many-body description often used in applications, for example, in phase transformation physics, statistical theory of turbulence, granular media dynamics etc, is the kinetic equation for the time evolution of the ‘relevant degrees of freedom’ distribution function. This equation is derived making strict assumptions about the nature of the fluctuations in the system—the underlying stochastic process performed by the relevant system degrees of freedom in the full phase space of the physical model. The equations of motion for the coarse-grained variables describing the meso- or macroscopic system properties are obtained from the kinetic equation by one of the known procedures, which are well-justified and understood, and occasionally are just a heuristic chain of semi-mathematical operations.

In this paper, we have discussed a novel approach to the description of dissipative systems dynamics, which is purely algebraic. Instead of making series of assumptions on the level of microscopic physics of the problem we assume the existence of certain algebraic structure, akin to that used in the Hamiltonian formulation of classical dynamics which permits us to derive meso- or macroscopic dissipative equations directly from the system-free energy. The basic ingredient of that procedure, the semimetric Leibniz bracket for dynamics variables over the whole phase space, is postulated according to our knowledge about dissipative processes. With knowledge of the Poissonian structure for the non-dissipative part of the system dynamics, the symmetries of the problem and using Dirac machinery [1] one can easily derive different Dirac structures, which are otherwise hard to postulate, describing dissipative dynamics. One may use the algorithm proposed here to design a variety of dissipative dynamical systems with the required conservative observables.

As shown in this paper, this permits us to build up the metriplectic dynamics scenario for several non-trivial systems: classical particle physics, many spins dynamics, rigid body dynamics, compressible and incompressible viscous fluid dynamics and some quantum mechanical problems. Several other applications, notably the relativistic charged particle systems which can also be formulated within the metriplectic scenario, have been discussed previously [17, 18]. The quantum applications are of particular interest in view of some similarity between the metriplectic approach and the Lindblads construction [19]—the standard tool in dissipative quantum system analysis. We expect to comment on the connection between both these approaches in a forthcoming publication.

Appendix

A symmetric Leibniz bracket which satisfies the SJ identity is called an SP bracket. The tensor G which generates the SJ bracket is called the *SP tensor*. In the local coordinates

$$\forall_{i,j=1,2,\ldots,N}\colon 0=\sum_{k=1}^{N}\left[G^{kj}\frac{\partial G^{ii}}{\partial z^{k}}-G^{ki}\frac{\partial G^{ij}}{\partial z^{k}}\right]. \tag{A.1}$$

Hence, the SP bracket and SP tensor are symmetric analogues to the Poisson bracket and Poisson tensor, respectively. In particular, each symmetric tensor where G_{ij} are constants (i.e. do not depend on points) generates a SP bracket. For a non-trivial example of a SJ tensor, see example A.1.

SP manifold is a pair (M, G) where G is a symmetric SP tensor of the type (2, 0).

Example A.1. It is easy to see that $G=\sum_{i,j=1}^{N} z_i z_j \frac{\partial}{\partial z_i}\otimes\frac{\partial}{\partial z_j}$ is a SP tensor. Hence, $(\mathbf{R}^N, G)$ is a SP manifold.

Similarly, with the concept of SJ identity one can define SL algebra, SP algebra which are symmetric analogues of Lie algebra and Poisson algebra, respectively. To prove that there is no other symmetric Jacobi identity we need a few elementary algebraic concepts. By an *identity* of the algebra $\mathcal{F}$ we mean a polynomial P in some free algebra which is identically zero when the generators are replaced by any elements of $\mathcal{F}$. We are interested in the three-linear identities in which each term involves two pairs of brackets, i.e. of the form: $a{\prec}{\prec}f, g{\succ}, h{\succ} + b{\prec}{\prec}f, h{\succ}, g{\succ} + c{\prec}{\prec}g, h{\succ}, f{\succ} = 0$. Now we would like to prove the following result:

Proposition A.1. *There are exactly two types of three-linear identities for symmetric algebra in which each term involves two pairs of brackets:*

(a) *The SE identity (symmetric version of the Engel identity)*: $\forall f\in\mathcal{F}\colon {\prec}{\prec}f, f{\succ}, f{\succ}=0$.

(b) *The SJ identity (symmetric version of the Jacobi identity)*:

$$\forall f, g\colon {\prec}{\prec}f, f{\succ}, g{\succ} = {\prec}{\prec}f, g{\succ}, f{\succ}$$
$$\iff \forall f, g, h\colon 2{\prec}{\prec}f, g{\succ}, h{\succ} = [{\prec}{\prec}f, h{\succ}, g{\succ} + {\prec}{\prec}g, h{\succ}, f{\succ}].$$

Proof. Suppose that $a{\prec}{\prec}f, g{\succ}, h{\succ}+b{\prec}{\prec}f, h{\succ}, g{\succ}+c{\prec}{\prec}g, h{\succ}, f{\succ}=0$. For $f=g=h$ we have $(a+b+c){\prec}{\prec}f, f{\succ}, f{\succ}=0$, therefore the identity must be of the type (a) or $a+b+c=0$. If $a+b+c=0$, then for $f=h$ we have $(a+c){\prec}{\prec}f, g{\succ}, f{\succ}+b{\prec}{\prec}f, f{\succ}, g{\succ}=0$, i.e. it must be of the type (b), since $a+c=-b$. □

Similarly, one can prove that there are exactly two three-linear identities in which each term involves two pairs of brackets for antisymmetric algebra:

(a) The Engel identity:

$$\forall f, g\colon \{\{f, g\}, f\}=0 \iff \forall f, g, h\colon \{\{f, g\}, h\}+\{\{h, g\}, f\}=0.$$

(b) The Jacobi identity:

$$\forall f, g, h\colon \{\{f, g\}, h\}+\{\{g, h\}, f\}+\{\{h, f\}, g\}=0.$$

Even the Jacobi identity and SJ identity are formally similar as we have seen above, but there also exists a fundamental difference between them. Indeed, the former is equivalent to $X_h\{f, g\}=\{X_h f, g\}+\{f, X_h g\}$, i.e. Hamiltonian vector fields act as derivations for the Poisson bracket, but for the latter $2X_h^{\mathrm{D}}{\prec}f, g{\succ}={\prec}X_h^{\mathrm{D}}f, g{\succ}+{\prec}f, X_h^{\mathrm{D}}g{\succ}$. In other words, the

Jacobi identity manifests some basic features like cyclicity, derivation and signature property while the SJ identity does not.

Furthermore, the most natural symmetric bracket $\prec A, B\succ = \frac{AB+BA}{2}$ does not satisfy the SJ identity, in general. Hence, there is no natural representation for SP algebras.

References

[1] Dirac P A M 1950 *Can. J. Math* **2** 129
Dirac P A M 1958 *Proc. R. Soc.* A **246** 326
Dirac P A M 1964 *Lecture Notes on Quantum Mechanics* (New York: Yeshiva University)
[2] Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetries* (Heidelberg: Springer) p 226
[3] Bhaskara K H and Viswanath K 1988 *Poisson Algebras and Poisson Manifolds* (London: Longman Scientific and Technical)
[4] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: a Modern Perspective* (New York: Wiley) ch 9
[5] Deriglazov A A, Galajinsky A V and Lyakhovich S L 1996 *Nucl. Phys.* B **473** 245
[6] Ferrari F and Lazziezzera I 1997 *Phys. Lett.* B **395** 250
[7] Nguyen S Q H and Turski Ł A 2001 *Physica* A **272** 48
Nguyen S Q H and Turski Ł A *Physica* A **290** 431
[8] Enz C P and Turski Ł A 1979 *Physica* A **96** 369
Turski Ł A 1990 *Continuum Models and Discrete Systems* vol. 1 ed G A Maugin (London: Longman)
Turski Ł A 1996 *Springer Lectures in Physics* vol. 4777 ed Z Petru, J Przystawa and K Rapcewicz (New York: Springer)
[9] Morrison P J 1986 *Physica* D **18** 410
[10] Enz C P 1977 *Physica* A **89** 1
[11] Gisin N 1982 *J. Phys.* A **14** 2259
Gisin N 1981 *Physica* A **11** 364
[12] Hołyst J A and Turski Ł A 1986 *Phys. Rev.* B **34** 1937
[13] Nguyen S Q H and Turski Ł A in preparation
[14] Nguyen S Q H and Turski Ł A in preparation
[15] Weinstein A 1983 *J. Diff. Geom.* **18** 523
[16] Śniatycki J 1974 *Ann. Inst. H Poincaré* **20** 365
[17] Białynicki-Birula I, Turski Ł A and Hubbard J C 1984 *Physica* A **128** 509
[18] Turski Ł A and A N Kaufman 1987 *Phys. Lett.* A **120** 331
[19] Lindblad G 1976 *Commun. Math. Phys.* **48** 119